Math 155, *Lecture Notes- Bonds*

Name_____________

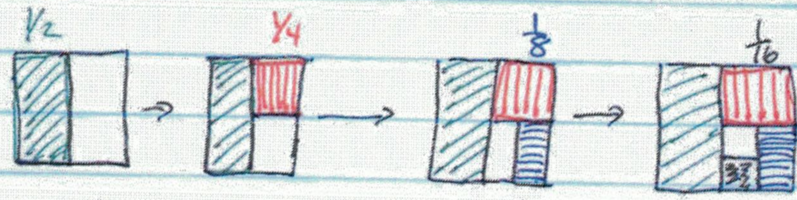
Section 9.2 Series and Convergence

Usually, we study infinite sequences in order to study "infinite summations." That is we want to consider what happens when we add tighter an infinite number of terms from a sequence. In this section we will begin our study of **infinite series.**

Ex. 1:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$



If we keep on adding half of what we added previously, what will happen?

To find the sum of an infinite series, we will consider the corresponding sequence of partial sums listed below:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4$$
$$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit S is called the **sum of the series.**

$$S = a_1 + a_2 + \cdots + a_n + \cdots$$

If $\{S_n\}$ diverges, then the series **diverges.**

$\{S_n\} \to S$ means

$$\lim_{n\to\infty} S_n = S$$

$$S_n = \sum_{i=1}^{n} a_i$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = S$$

more **Ex. 1:**

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

$$\begin{aligned} S &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} \frac{2^n - 1}{2^n} \\ &= \lim_{n\to\infty}\left(\frac{2^n}{2^n} - \frac{1}{2^n}\right) \\ &= \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) \\ &= 1 \end{aligned}$$

Since the sequence of partial sums converges to 1, $\{S_n\} \to 1$, we say that

$$S = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Geometric Sequences and Series

– Geometric Sequences and Series –

– A sequence $\{b_1, b_2, b_3, \ldots\ldots\}$ is called a geometric sequence if for every $n \in \mathbb{N}$,

$$\frac{b_{n+1}}{b_n} = r \quad \text{where } r \text{ is a constant.}$$

This constant r is called the ratio of the geometric sequence.

Ex. 2:

Example: $3, 6, 12, 24, 48, \ldots\ldots$

$$\frac{6}{3} = 2, \quad \frac{12}{6} = 2, \quad \frac{24}{12} = 2, \quad \frac{48}{24} = 2$$

So, $r = 2$

Theorem: If $\{b_n \mid n \in \mathbb{N}\}$ is a geometric sequence, then

$$b_n = b_1 \cdot r^{n-1}$$

, where r is the ratio of the sequence.

Ex. 3:

Example: $b_1 = 3$, $r = 2$

$$b_4 = b_1 r^{4-1}$$
$$b_4 = b_1 r^3$$
$$b_4 = (3) \cdot (2)^3$$
$$b_4 = 3 \cdot 8$$
$$b_4 = 24$$

Consider, $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots$, $a \neq 0$

$\sum_{n=0}^{\infty} ar^n$ is a geometric series with ratio r

Fact: $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$, $r \neq 1$

Proof:

let $S_n = \sum_{i=0}^{n-1} r^i = \underbrace{\overset{i=0}{1} + \overset{i=1}{r} + \overset{i=2}{r^2} + \cdots + \overset{i=n-2}{r^{n-2}} + \overset{i=n-1}{r^{n-1}}}_{n \text{ terms}}$

$$r \cdot S_n = r(1 + r + r^2 + \cdots + r^{n-2} + r^{n-1})$$
$$= r + r^2 + r^3 + \cdots + r^{n-1} + r^n$$

Consider $S_n - rS_n = S_n(1-r)$

$(1+\cancel{r}+\cancel{r^2}+\cdots+\cancel{r^{n-2}}+\cancel{r^{n-1}}) - (\cancel{r}+\cancel{r^2}+r^3+\cdots+\cancel{r^{n-1}}+r^n) = S_n(1-r)$

$1 - r^n = S_n(1-r)$

So, $\boxed{S_n = \dfrac{1-r^n}{1-r}}$

THEOREM 9.6 Convergence of a Geometric Series

A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

Proof: Let $S_n = \sum_{i=0}^{n-1} ar^i$

$= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$

$= a(1 + r + r^2 + \cdots + r^{n-2} + r^{n-1})$

$S_n = a\left(\dfrac{1-r^n}{1-r}\right)$

Case 1: If $|r| > 1$, then $r^n \to \infty$ as $n \to \infty$

So, $\lim_{n\to\infty} S_n = \infty$

Case 2: If $|r| = 1$, then

$S_n = \underbrace{a + a + a + a + a + a + \cdots + a + a}_{n \text{ term}}$, subcase 1, $S_n = n \cdot a$, or

subcase 2, $S_n = a - a + a - a + a - a + a \ldots$

subcase 1: $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} n\,a = \infty$

subcase 2: $\lim\limits_{n\to\infty} S_n$ does not exist due to oscillation

Case 3: If $0<|r|<1$, then $\lim\limits_{n\to\infty} r^n = 0$, and

we have $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} a\left(\dfrac{1-\overset{\to 0}{r^n}}{1-r}\right)$

$= \boxed{\dfrac{a}{1-r}}$

↑ we will use the result through the rest of the chapter.

Ex. 4: Find the sum: $\sum\limits_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n = \sum\limits_{n=0}^{\infty} a \cdot r^n = \dfrac{a}{1-r}$, $a=6$, $r=\frac{4}{5}$

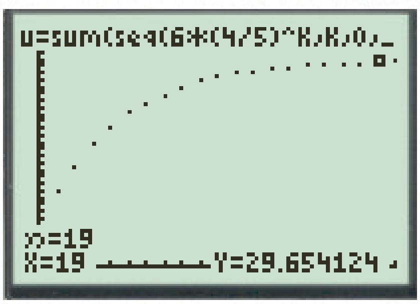
$= \dfrac{(6)}{1-\left(\frac{4}{5}\right)}$

$= \left[\dfrac{\frac{6}{1}}{1-\frac{4}{5}}\right] \cdot \left[\dfrac{\frac{5}{1}}{\frac{5}{1}}\right]$

$= \dfrac{30}{5-4}$

$= \dfrac{30}{1}$

$= 30$




```
sum(seq(6*(4/5)^
K,K,0,100))
                30
```

```
 Plot1 Plot2 Plot3
 nMin=0
\u(n)=sum(seq(6*
(4/5)^K,K,0,n))
 u(nMin)=
\v(n)=
 v(nMin)=
\w(n)=
```

```
(6)/(1-(4/5))▸Fr
ac
                30
```

n	u(n)
8	25.973
9	26.779
10	27.423
11	27.938
12	28.351
13	28.681
14	28.944

n=14

Ex. 5: Find the sum: $\sum_{n=0}^{\infty} 6\left(-\frac{4}{5}\right)^n = \sum_{n=0}^{\infty} 6\left(\frac{-4}{5}\right)^n = \frac{a}{1-r}$, $a=6$, $r=\frac{-4}{5}$

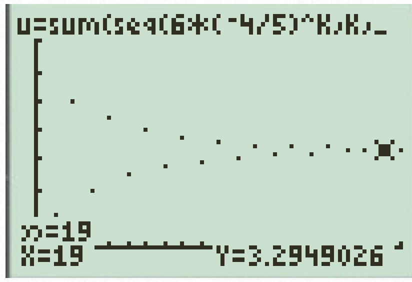
$= \frac{(6)}{1-\left(\frac{-4}{5}\right)}$

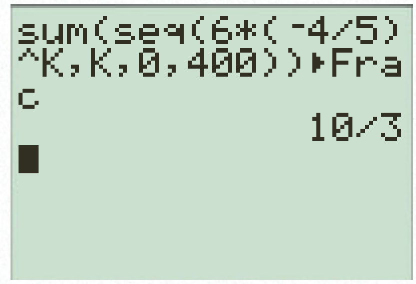
$= \left[\frac{\frac{6}{1}}{1+\frac{4}{5}}\right] \cdot \left[\frac{\frac{5}{1}}{\frac{5}{1}}\right]$

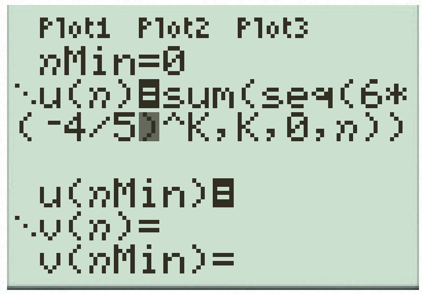
$= \frac{30}{5+4}$

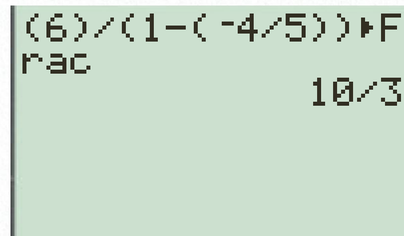
$= \frac{30}{9}$

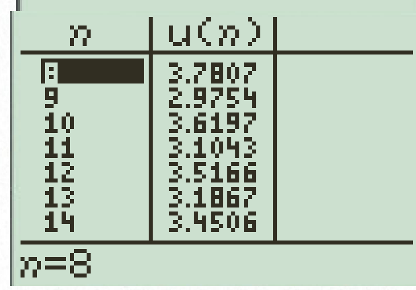
$= \frac{10}{3}$











Ex. 6: Find the sum: $0.\bar{9} = 0.9999\ldots = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots$

$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1{,}000} + \frac{9}{10{,}000} + \ldots$

$a = \frac{9}{10}$, $r = \frac{a_2}{a_1} = \frac{\frac{9}{100}}{\frac{9}{10}} = \frac{9}{100} \cdot \frac{10}{9} = \frac{1}{10}$ ✓

$= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^n$

$= \frac{a}{1-r}$

$= \frac{\left(\frac{9}{10}\right)}{1-\left(\frac{1}{10}\right)}$

$= \left[\frac{\frac{9}{10}}{1-\frac{1}{10}}\right] \cdot \left[\frac{\frac{10}{1}}{\frac{10}{1}}\right]$

$= \frac{9}{10-1} = \frac{9}{9} = \underline{1}$ ✓

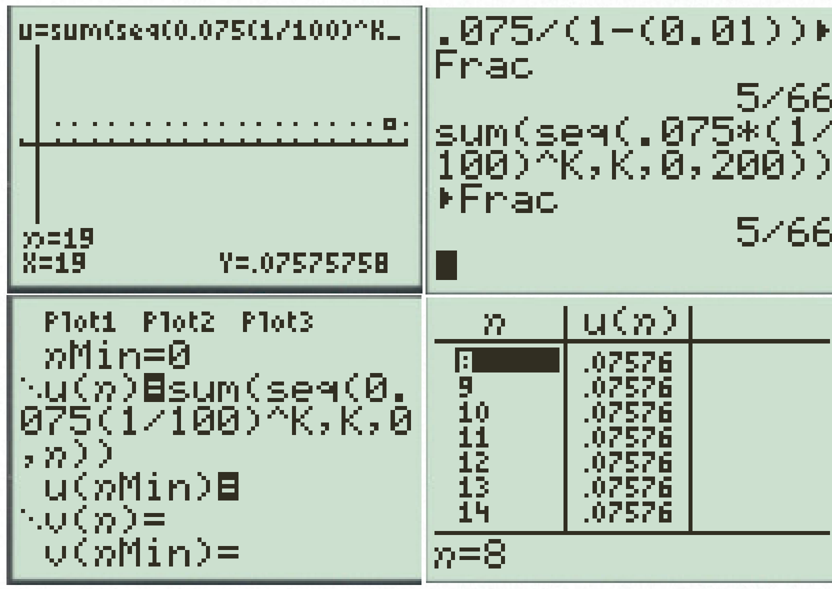
Ex. 7: Find the sum: $0.0\overline{75} = 0.0757575\ldots = 0.075757575\ldots$

$= 0.075 + 0.00075 + 0.0000075 + \ldots$

$= \frac{75}{1{,}000} + \frac{75}{100{,}000} + \frac{75}{10{,}000{,}000} + \ldots$

$a = \frac{75}{1{,}000}$, $r = \frac{a_2}{a_1} = \frac{\frac{75}{100{,}000}}{\frac{75}{1{,}000}} = \frac{75}{100{,}000} \cdot \frac{1{,}000}{75} = \frac{1}{100}$ ✓

$= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{75}{1{,}000}\left(\frac{1}{100}\right)^n$



$= \frac{a}{1-r}$

$= \frac{\left(\frac{75}{1{,}000}\right)}{1-\left(\frac{1}{100}\right)}$

$= \left[\frac{\frac{75}{1{,}000}}{1-\frac{1}{100}}\right] \cdot \left[\frac{\frac{1000}{1}}{\frac{1{,}000}{1}}\right]$

$= \frac{75}{1000-10}$

$= \frac{75}{990}$

$= \frac{\cancel{3}\cdot\cancel{5}\cdot 5}{2\cdot 3\cdot\cancel{3}\cdot\cancel{5}\cdot 11}$

$= \frac{5}{66}$ ✓

Ex. 8: (a) Find all values of x for which $\sum_{n=0}^{\infty} 4\left(\frac{x-3}{4}\right)^n$ converges.

(b) For these values of x, write the sum of the series.

(a) $\sum_{n=0}^{\infty} 4\left(\frac{x-3}{4}\right)^n = \sum_{n=0}^{\infty} ar^n$, where $a=4$ & $r=\frac{x-3}{4}$.

This geometric series converges when $|r|<1$.

solve for x:

$$\left|\frac{x-3}{4}\right|<1$$

$$4\cdot\left|\frac{x-3}{4}\right|<4\cdot 1$$

$$|x-3|<4$$

$$-4<x-3<4$$

$$-4+3<x-3+3<4+3$$

$$-1<x<7$$

(b) If $x\in(-1,7)$, then

$$\sum_{n=0}^{\infty} 4\left(\frac{x-3}{4}\right)^n = \frac{a}{1-r}$$

$$= \frac{(4)}{1-\left(\frac{x-3}{4}\right)}$$

$$= \left[\frac{\frac{4}{1}}{1-\left(\frac{x-3}{4}\right)}\right]\cdot\left[\frac{\frac{4}{1}}{\frac{4}{1}}\right]$$

$$= \frac{16}{4-(x-3)}$$

$$= \frac{16}{4-x+3}$$

$$= \frac{16}{7-x}$$

THEOREM 9.7 Properties of Infinite Series

If $\Sigma a_n = A$, $\Sigma b_n = B$, and c is a real number, then the following series converge to the indicated sums.

1. $\sum_{n=1}^{\infty} ca_n = cA$
2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

<u>*Telescoping Series-*</u> *"sometimes you get lucky"*

A **telescoping series** is a special form that "collapses" like an old-fashioned telescope.

$$S = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \dots$$

Since the "interior" terms cancel, we can consider the nth partial sum:

$$S_n = b_1 - b_{n+1}$$

If the series converges, we can use this nth partial sum to find the sum of the series by taking the limit:

$$S = b_1 - \lim_{n\to\infty} b_{n+1}$$

Ex. 9: Find the sum: $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$, Re-write using Partial Fraction Decomposition

$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$

$1 = A(n+2) + Bn$ (Basic Equation)

choose: $n=0$

$1 = A[(0)+2] + B(0)$

$1 = 2A$

$\frac{1}{2} = A$

$n=-2$

$1 = \frac{1}{2}\cdot[(-2)+2] + B(-2)$

$1 = \frac{1}{2}\cdot[0] - 2B$

$1 = -2B$

$-\frac{1}{2} = B$

So, $\frac{1}{n(n+2)} = \frac{\frac{1}{2}}{n} + \frac{-\frac{1}{2}}{n+2} = \frac{1}{2n} - \frac{1}{2(n+2)}$

Ex. 9: Continued

Since $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left[\frac{1}{2n} - \frac{1}{2(n+2)}\right]$

$= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots$

$n=1$ $n=2$ $n=3$ $n=4$ $n=5$

Don't cancel , telescoping form

So, $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left[\frac{1}{2n} - \frac{1}{2(n+2)}\right]$

$b_1 = \frac{1}{2} + \frac{1}{4}$

$= \lim_{n\to\infty} S_n$

$b_{n+1} = \frac{1}{2(n+1)} - \frac{1}{2[(n+1)+2]}$

$= \frac{1}{2n+2} - \frac{1}{2n+6}$

$= \lim_{n\to\infty} \left[b_1 - b_{n+1}\right]$

$= \lim_{n\to\infty} \left[\left(\frac{1}{2} + \frac{1}{4}\right) - \left(\frac{1}{2n+2} - \frac{1}{2n+6}\right)\right]$

$= \frac{1}{2} + \frac{1}{4} + 0$

$= \frac{3}{4}$

Series convergence implies that the nth term tends to zero. Here are two theorems about this:

THEOREM 9.8 Limit of nth Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

THEOREM 9.9 nth-Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex. 10: Find the sum: $\sum_{n=0}^{\infty} (1.075)^n$

This is a geometric series with $a=1$ & $r=1.075$

$$\sum_{n=0}^{\infty} (1.075)^n = \sum_{n=0}^{\infty} 1\cdot(1.075)^n = \sum_{n=0}^{\infty} ar^n$$

since $|r| \geq 1$, the series diverges.

Ex. 11: Find the sum: $\sum_{n=1}^{\infty} \frac{2^n}{100}$

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2^n}{100}$, where $a_n = \frac{2^n}{100}$.

Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{100}$

$= \frac{1}{100} \cdot \lim_{n\to\infty} (2^n)$

$= \frac{1}{100} \cdot \infty$

$= \infty$

Since $\lim_{n\to\infty} a_n \neq 0$, the nth Term Test for Divergence tells us that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2^n}{100}$ diverges.

Ex. 12: Find the sum: $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+1}{2n-1}$, where $a_n = \frac{n+1}{2n-1}$.

Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{2n-1}$

$$= \lim_{n\to\infty} \left[\frac{\frac{n+1}{1}}{\frac{2n-1}{1}}\right] \cdot \left[\frac{\frac{1}{n}}{\frac{1}{n}}\right]$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{1}{n}}$$

$$= \frac{1+0}{2-0}$$

$$= \frac{1}{2}$$

Since $\lim_{n\to\infty} a_n \neq 0$, the nth Term Test for Divergence tells us that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+1}{2n-1}$ diverges.